MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 6

Recall that $(T, <)$ is a tree if $<$ is a transitive well founded ordering, such that for every $x \in T$, the predecessors of x are a well ordered set, i.e. it has an order type. Denote this order type by $o(x)$. The height of the tree, $ht(T) = \sup_{x \in T} o(x)$, and for every $\alpha < ht(T)$, the α-th level of T is $T_\alpha = \{x \in T \mid o(x) = \alpha\}$. T is a κ-tree if it has height κ and levels of size less than κ. A branch though T is a maximal linearly ordered subset of T. We will write $x \perp y$ to denote that x and y are incomparable.

Let b be an unbounded branch through a tree T. Then:

- for all $\alpha < ht(T)$, $|b \cap T_\alpha| = 1$,
- if $x < y$ and $y \in b$, then $x \in b$,
- if $x \perp y$ and $y \in b$, then $x \notin b$,
- if $y \in b \cap T_\alpha$, then $pred(y) := \{x \in T \mid x < y\} = b \cap \bigcup_{\beta < \alpha} T_\beta$.

Lemma 1. *Suppose that $T \in V$, $\mathbb{P}$ is a poset, such that if G is $\mathbb{P}$-generic, than in $V[G]$, there is an unbounded branch through T. Let $\dot{b}$ be a $\mathbb{P}$-name, such that $1_{\mathbb{P}} \Vdash \dot{b}$ is an unbounded branch though T. Then,*

1. *If p, q are compatible, $\alpha < \beta$, $p \Vdash x \in \dot{b} \cap T_\alpha$, and $q \Vdash y \in \dot{b} \cap T_\beta$, then $x <_T y$.*
2. *If $p \Vdash x \in \dot{b} \cap T_\alpha$, $q \Vdash y \in \dot{b} \cap T_\alpha$, and $x \neq y$, then p and q are incompatible.*
3. *If $p \Vdash y \in \dot{b}$ and $x <_T y$, then $p \Vdash x \in \dot{b}$.*
4. *If $\alpha < ht(T)$ and $p \in \mathbb{P}$, then there is $q \leq p$ and $x \in T_\alpha$, such that $q \Vdash x \in \dot{b}$.*

Proof. (1): Let r be a common extension of p, q. Since 1 forces that $\dot{b}$ is a branch, r forces that $\dot{b}$ is linearly ordered. Also, $r \leq p$, so $r \Vdash x \in \dot{b}$; and $r \leq q$, so $r \Vdash y \in \dot{b}$. Then x, y must be comparable. Since $\alpha < \beta$, then $x <_T y$.

(2): Suppose for contradiction that r is a common extension of p, q. Then $r \Vdash x, y \in \dot{b} \cap T_\alpha$. But distinct notes of the same level are incomparable. Contradiction with the fact that r forces that b is linearly ordered.

(3): One of the properties of being a branch is that it is closed under predecessors. Since $\dot{b}$ is forced to be a branch by the empty condition, p forces that $\dot{b}$ is closed under predecessors.

(4): p forces that $\dot{b}$ is unbounded. I.e. $p \Vdash (\forall \beta < ht(t)) \dot{b} \cap T_\beta \neq \emptyset$. So, $p \Vdash \dot{b} \cap T_\alpha \neq \emptyset$. So, there is $x \in T_\alpha$ and $q \leq p$, such that $q \Vdash x \in \dot{b}$.

$\square$

Lemma 2. *Same assumptions as above. Suppose in addition, that there are no branches through T in V. Then for every p, for every $\alpha < ht(T)$, there is $\beta \geq \alpha$, conditions q_1, q_2 stronger than p and distinct nodes $x, y \in T_\beta$, such that $q_2 \Vdash y \in \dot{b}$ and $q_1 \Vdash x \in \dot{b}$. Note that q_1 and q_2 must be incompatible.*

Proof. Suppose otherwise. Let $e = \{x \in T \mid (\exists q \leq p) q \Vdash x \in \dot{b}\}$. Note that by the above lemma, for every $\beta < ht(T)$, $e \cap T_\beta \neq \emptyset$, and also that e is closed under predecessors. By our assumption for every $\beta > \alpha$, $|e \cap T_\beta| = 1$. We claim that e is an unbounded branch. It suffices to show that any two elements in e are comparable. Suppose that $x, y \in e$. Let $\beta < \gamma$ be such that $x \in T_\beta, y \in T_\gamma$. Let q, r be stronger than p, such that $q \Vdash x \in \dot{b}, r \Vdash y \in \dot{b}$. Let $\gamma' > \max(\gamma, \alpha)$. By the last item of the previous lemma, there is $r' \leq r$ and $z \in T_{\gamma'}$, such that $r' \Vdash z \in \dot{b}$. Note that $z \in e$ and by item (1) of the last lemma, $y < z$.

Since q forces that $x \in \dot{b}$, $\dot{b}$ is unbounded, and linearly ordered, there is some $q' \leq q$ and z' with $x < z'$, such that $q' \Vdash z' \in \dot{b} \cap T_{\gamma'}$. But then $z' \in e$ and since $|e \cap T_{\gamma'}| = 1$, we get $z' = z$. So $x < z$. But then $x < y$.

It follows that e is an unbounded branch though T, which is a contradiction with the assumption that there are no branches through T in V.

□

Next we discuss forcings that cannot add new branches.

Definition 3. $\mathbb{P}$ *is κ-Knaster if for every set $\{p_\alpha \mid \alpha < \kappa\}$ of conditions, there is an unbounded $I \subset \kappa$, such that $\{p_\alpha \mid \alpha \in I\}$ are pairwise compatible.*

Note that being κ-Knaster, implies the κ-chain condition. Also, by the Δ-system lemma, the Cohen poset $Add(\tau, \lambda)$ is τ^+-Knaster for any λ. In particular, $Add(\omega, \lambda)$ is ω_1-Knaster.

Lemma 4. *Suppose that T is a tree of height κ and $\mathbb{P}$ is a κ-Knaster forcing. Then forcing with $\mathbb{P}$ does not add new branches.*

Proof. Suppose otherwise. Let $p \in \mathbb{P}$ be such that $p \Vdash \dot{b}$ is a branch though T. For every $\alpha < ht(T)$, let p_α and x_α be such that $p_\alpha \Vdash x_\alpha \in T_\alpha \cap \dot{b}$. Since $\mathbb{P}$ is κ-Knaster, there is unbounded $I \subset \kappa$, such that $\langle p_\alpha \mid \alpha \in I \rangle$ are pairwise compatible. We claim that $\langle x_\alpha \mid \alpha \in I \rangle$ generate an unbounded branch. First note that by one of the above lemmas, for every $\alpha < \beta$, $\alpha, \beta \in I$, $x_\alpha < x_\beta$. So $e := \{x \mid (\exists \alpha \in I) x < x_\alpha\}$ is a branch through T in V. Contradiction.

□

In particular, since the poset $Add(\omega, \kappa)$ is ω_1-Knaster, it cannot add a branch though a a tree of height ω_1. Note that we did not assume above that T is a κ-tree, i.e. the levels of the tree above may have size κ.